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Fixed points of contractive and Geraghty contraction mappings under the influence of altering distances

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Abstract
In this paper, we prove the existence of fixed points of contractive and Geraghty contraction maps in complete metric spaces under the influence of altering distances. Our results extend and generalize some of the known results.
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I. Introduction and Preliminaries

In a complete metric space, Geraghty[4] established a criteria for the sequence of Picard iterations defined by $x_0 \in X$, $x_n = Tx_{n-1}$, $n = 1, 2, 3, \ldots$ to be Cauchy for contractive mappings. If it is Cauchy, it is easy to see that it converges to a unique fixed point of T in X and proved necessary and sufficient condition for a sequence of iterates to be convergent.

Notation: Throughout this paper,

$\Psi = \{\psi: R^+ \to R^+ / \psi$ is non-decreasing and $\psi(t) = 0 \Leftrightarrow t = 0\}$,

$S = \{\beta: [0, \infty) \to [0, 1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$.

$\Delta_n = \frac{d(T(xh(n)),T(xk(n)))}{dn}$,

$$\Delta_n^{\psi} = \frac{\psi(d(T(xh(n)),T(xk(n))))}{\psi(dn)}$$

and $d_n = d(x_{h(n)}, x_{k(n)})$

for any sequence $\{x_n\}$ in X and subsequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ of $\{x_n\}$.

We write $R^+ = [0, \infty)$.

Definition 1.1. Let (X, d) *be* metric space and let $T: X \to X$ be a selfmap. We say that T is contractive mapping, if

$$d(T(x), T(y)) < d(x, y) \qquad (1.1.1)$$

for all x, y in X *with* $x \neq y$.

Definition 1.2. Let (X, d) be a metric space and let $T: X \to X$ be a selfmap. We say that T is contraction mapping, if there exists $k \in [0, 1)$ such that

$$d(T(x), T(y)) \leq k\, d(x, y) \qquad (1.2.1)$$

for all x, y in X.

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Geraghty [4] proved the following theorems.

Theorem 1.3. (Geraghty)([4] Theorem 1.1.) Let (X, d) be a complete metric space. Let $T: X \to X$ *be a selfmap on* X *such that*

$$d(Tx, Ty) < d(x, y) \qquad (1.3.1)$$

for all $x, y \in X$ *with* $x \neq y$.

Let $x_0 \in X$ and set $x_n = Tx_{n-1}$ for $n > 0$. Then $x_n \to x_\infty$ in X, with x_∞ a unique fixed point of T, if and only if for any two sub sequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that $\Delta_n \to 1$ only if $d_n \to 0$.

Theorem 1.4. (Geraghty) ([4] Theorem 1.3.) Let $T: X \to X$ be a contraction on a complete metric space (X, d). Let $x_0 \in X$ and set $x_n = Tx_{n-1}$ for $n > 0$. Then $x_n \to x_\infty$, where x_∞ is a unique fixed point of T in X, if and only if there exists $\beta \in S$ such that for all n, m,

$$d(Tx_n, Tx_m) \leq \beta(d(x_n, x_m)).d(x_n, x_m). \qquad (1.4.1)$$

Definition 1.5. Altering distance function [6] :

A function $\varphi : R^+ \to R^+$ is said to be an *altering distance function* if the following conditions hold:

(i) φ is continuous,

(ii) φ is non-decreasing and

(iii) $\varphi(t) = 0$ if and only if $t = 0$.

We denote the class of all altering distance functions $\varphi: R^+ \to R^+$ by Φ. It may be noted that Φ is a proper sub class of Ψ.

For more literature on the existence of fixed points of different contraction conditions involving altering distance functions, we refer to [[3], [6], [7]].

(1.2.1)

In 2013, Babu and Subhashini proved the following results by taking

$\Phi_1 = \{\varphi_1: [0,\infty) \to [0,\infty) / \varphi_1$ is continuous and $\varphi_1(t) = 0 \Leftrightarrow t = 0\}$

Theorem 1.6. (Babu and Subhashini [2]): Let T be a self map on a complete metric space (X, d). Assume that there exist a $\varphi_1 \in \Phi_1$ with

$$\varphi_1(d(Tx, Ty)) < \varphi_1(d(x, y)) \quad (1.6.1)$$

for all x, y in X with $x \neq y$.
Let $x_0 \in X$, and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \ldots$.
Then $x_n \to z$ in X, z is a unique fixed point of T if and only if for any two subsequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that $\Delta_n^{\varphi_1} \to 1$ only if $d_n \to 0$.

Where $\Delta_n^{\varphi_1} = \dfrac{\varphi_1(d(Tx_{h(n)}, Tx_{k(n)}))}{\varphi_1(d_n)}$, and $d_n = d(x_{h(n)}, x_{k(n)})$.

Theorem 1.7. (Babu and Subhashini [2]) : Let (X, d) be a complete metric space and T be a self map on X. Assume that there exist $\varphi_1 \in \Phi_1$ and $k \in [0, 1)$ such that

$$\varphi_1(d(Tx, Ty)) \leq k\, \varphi_1(d(x, y)) \quad (1.7.1)$$

for all $x, y \in X$,
Let $x_0 \in X$ and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3 \ldots$
Then $x_n \to z$, z is a unique fixed point of T in X if and only if there exists $\beta \in S$ such that for all $n, m \in N$.

$$\varphi_1(d(Tx_n, Tx_m)) \leq \beta(\varphi_1(d(x_n, x_m)))\, \psi(d(x_n, x_m)) \quad (1.7.2)$$

In 2012, Gordji, Ramezani, Cho and Pirbavafa [5] proved the following theorem in partially ordered metric spaces by using an element $\psi \in \Psi$ along with the additional assumptions of continuity and sub-additivity.

Theorem 1.8. (Gordji et al. [5]) Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X \to X$ be a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \preccurlyeq T(x_0)$. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$, ψ is continuous and sub-additive such that

$$\psi(d(T(x), T(y))) \leq \beta(\psi(d(x, y)))\psi(d(x, y)) \quad (1.8.1)$$

for all $x, y \in X$ with $x \succcurlyeq y$.
Assume that either
(i) T is continuous (*or*)
(ii) X is such that if an increasing sequence $\{x_n\}$ converges to x then $x_n \preccurlyeq x$ for each $n \geq 1$ holds.
Then T has a fixed point.
Further, if for each $x, y \in X$, there exists $z \in X$ which is comparable to x and y.
Then T has a unique fixed point in X.
The following lemma, which we use in the next section, can be easily established.

Lemma 1.9. [1] Let (X, d) be metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ and $d(xm(k), xn(k)) \geq \varepsilon$.
For each $k > 0$, corresponding to $m(k)$, we can choose $n(k)$ to be the smallest integer such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$ and $d(x_{m(k)}, x_{n(k)-1}) < \varepsilon$.
It can be shown that the following identities are satisfied.

(i) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \varepsilon$,

(ii) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$,

(iii) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon$ and

(iv) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon$.

II. Main results

In this section, we first prove the following Theorem (which is a variant of Theorem 1.3, without using the continuity of altering distance function).

Theorem 2.1. Let T be a selfmap on a complete metric space (X, d). Assume that there exists $\psi \in \Psi$ such that

$$\psi(d(T(x), T(y))) < \psi(d(x, y)) \quad (2.1.1)$$

for all x, y in X with $x \neq y$.
Let $x_0 \in X$ and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3 \ldots$
Suppose for any two subsequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that

$$\Delta_n^{\psi} \to 1 \text{ only if } \psi(d_n) \to 0. \quad (C)$$

Then $\{x_n\}$ is a Cauchy sequence and hence converges say to z and z is the unique fixed point of T in X.

Proof. From (2.1.1), we have
$\psi(d(T(x), T(y))) < \psi(d(x, y))$ for all x, y in X with $x \neq y$
$\Rightarrow d(T(x), T(y)) < d(x, y)$ for all x, y in X with $x \neq y$
$\Rightarrow d(T(x), T(y)) \leq d(x, y)$ for all x, y in X.
Therefore T is continuous.
By (2.1.1), clearly, if T has a fixed point, then it is unique.
Assume that condition (C) holds.
i.e., $\Delta_n^{\psi} \to 1 \Rightarrow \psi(d_n) \to 0$.
Let $x_0 \in X$.
We define $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for each $n = 1, 2, 3\ldots$
If $x_n = x_{n+1}$ for some n then x_n is a fixed point of f.
Without loss of generality, we assume that $x_n \neq x_{n+1}$ for each n.
We have

$$\psi(d(x_{n+1}, x_{n+2})) = \psi(d(Tx_n, Tx_{n+1})) < \psi(d(x_n, x_{n+1})). \quad (2.1.2)$$

Therefore $\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_n, x_{n+1}))$ and hence $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all n.
Thus it follows that $\{\psi(d(x_n, x_{n+1}))\}$ and $\{d(x_n, x_{n+1})\}$ are strictly decreasing sequences of positive real

numbers and so $\lim_{n\to\infty} \psi(d(x_n, x_{n+1}))$ exists and it is r (say). *i.e.,* $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = r \geq 0$.

Also $\lim_{n\to\infty} d(x_n, x_{n+1})$ exists and it is s (say).

i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = s \geq 0$.

We now show that $r = 0$ and $s = 0$.

Assume that $r > 0$, By choosing $h_n = n$ and $k_n = n + 1$ in (C).

$$\frac{\psi(d(Tx_{h(n)}, Tx_{k(n)}))}{\psi(d(x_{h(n)}, x_{k(n)}))} = \frac{\psi(d(Tx_n, Tx_{n+1}))}{\psi(d(x_n, x_{n+1}))} = \frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \to 1 \text{ as } n \to \infty$$

and hence from condition(C),

$\psi(d(x_n, x_{n+1})) \to 0$ as $n \to \infty$.

Therefore $r = 0$.

Suppose $s \neq 0$ that implies there exists N such that

$\frac{s}{2} < d(x_n, x_{n+1})\ \forall\ n \geq N.$

$\Rightarrow 0 < \psi(\frac{s}{2}) < \psi(d(x_n, x_{n+1}))$,

a contradiction $\forall n \geq N$.

Hence $s = 0$.

Now, we show that $\{x_n\}$ is Cauchy.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma (1.9) there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) > \varepsilon$.

For each $k > 0$, corresponding to $m(k)$, we can choose $n(k)$ to be the smallest integer such that $d(x_{m(k)}, x_{n(k)}) > \varepsilon$ and $d(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon$.

It can be shown that the following identities are satisfied.

(*i*) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$

(*ii*) $\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)-1}) = \varepsilon$,

(*iii*) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon$ and

(*iv*) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon$.

We have

$0 \leq \psi(\varepsilon)$ and hence $x_{m(k)-1} \neq x_{n(k)-1}$

$0 \leq \psi(\varepsilon) \leq \psi(d(x_{m(k)}, x_{n(k)}) = \psi(d(Tx_{m(k)-1}, Tx_{nm(k)-1})$

$<\psi(d(x_{m(k)-1}, x_{n(k)-1}))$ by(2.1.1)

Therefore $\frac{\psi(d(x_{m(k)}, x_{n(k)}))}{\psi(d(x_{m(k)-1}, x_{n(k)-1}))} < 1$.

Suppose $\frac{\psi(d(x_{m(k)}, x_{n(k)}))}{\psi(d(x_{m(k)-1}, x_{n(k)-1}))} \to 1$.

Then by condition (C)

$\psi(d(x_{m(k)-1}, x_{n(k)-1})) = 0$.

$\Rightarrow \psi(\varepsilon - 0) \leq \psi(d(x_{m(k)-1}, x_{n(k)-1})) = 0$.

Therefore $\psi(\varepsilon) = 0 \Rightarrow \varepsilon = 0$,

a contradiction.

Suppose $\frac{\psi(d(x_{m(k)}, x_{n(k)}))}{\psi(d(x_{m(k)-1}, x_{n(k)-1}))}$ does not tend to 1.

Then without loss of generality, we may suppose that there exist $0 < \alpha < 1$ such that

$\frac{\psi(d(x_{m(k)}, x_{n(k)}))}{\psi(d(x_{m(k)-1}, x_{n(k)-1}))} < \alpha$, for all k.

Therefore $0 \leq \psi(d(x_{n(k)}, x_{m(k)})) \leq \alpha\ \psi(d(x_{n(k)-1}, x_{m(k)-1}))$.

On letting $k \to \infty$, we get $0 \leq \psi(\varepsilon + 0) \leq \alpha\ \psi(\varepsilon + 0)$

$\Rightarrow \psi(\varepsilon + 0) = 0$.

Therefore $\varepsilon = 0$,

a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence in X, and since X is complete, there exists $z \in X$ such that $\lim_{k\to\infty} x_n = z$.

Now, we show that z is a fixed point of T.

Since T is continuous,

In this case,

we have $\lim_{k\to\infty} x_n = \lim_{k\to\infty} x_{n+1} = \lim_{k\to\infty} Tx_n = Tz$.

Therefore z is a fixed point of T in X and z is unique.

Theorem 2.2. Let (X, d) be a complete metric space and T be a continuous self map on X.

Let $x_0 \in X$ and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \ldots$.

Suppose that there exist $\psi \in \Psi$ and $\beta \in S$ such that for all $n, m \in N$.

$\psi(d(Tx_n, Tx_m)) \leq \beta(\psi(d(x_n, x_m)))\psi(d(x_n, x_m))$. (2.2.1)

Then $\{x_n\}$ is Cauchy, $\{x_n\}$ converges to z (say) and z is a unique fixed point of T in X.

Proof. We prove that $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not Cauchy.

Then as in the proof of Theorem 2.1, we can find sequences $\{m_k\}$ and $\{n_k\}$ satisfying the inequalities there in.

Then we have

$\psi(d(x_{n(k)}, x_{m(k)})) = \psi(d(Tx_{n(k)-1}, Tx_{m(k)-1}))$

$\leq \beta(\psi(d(x_{n(k)-1}, x_{m(k)-1}))(\psi(d(x_{n(k)-1}, x_{m(k)-1}))$ (2.2.2)

Suppose $\beta\ (\psi(d(x_{n(k)-1}, x_{m(k)-1})) \to 1$ then by hypothesis $\psi(d(x_{n(k)-1}, x_{m(k)-1})) \to 0$

that implies $\psi(d(x_{n(k)}, x_{m(k)})) \to 0$.

Now, $\varepsilon < d(x_{n(k)}, x_{m(k)})$.

Therefore $0 \leq \psi(\varepsilon) \leq \psi(d(x_{n(k)}, x_{m(k)})) \to 0$ and hence $\psi(\varepsilon) = 0$ that implies $\varepsilon = 0$,

a contradiction.

Hence without loss of generality, we may suppose that there exist $0 < \alpha < 1$ such that $\beta(\psi(d(x_{n(k)-1}, x_{m(k)-1})) < \alpha$ for infinitely many n.

Then from (2.2.1)

$0 \leq \psi(d(x_{n(k)}, x_{m(k)})) \leq \alpha\ \psi(d(x_{n(k)-1}, x_{m(k)-1}))$.

On letting $k \to \infty$, we get

$0 \leq \psi(\varepsilon + 0) \leq \alpha.\ \psi(\varepsilon + 0) \Rightarrow \psi(\varepsilon + 0) = 0 \Rightarrow \varepsilon = 0$,

a contradiction.
Therefore $\{x_n\}$ is a Cauchy sequence in X, and since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.
Now, we show that z is a fixed point of T.
Since T is continuous
$x_n \to z \Rightarrow Tx_n \to Tz$
$\Rightarrow x_{n+1} \to Tz$.
Therefore by the uniqueness of the limit, $z = Tz$ and hence z is a fixed point of T in X.
Suppose that z and w are two fixed points of T with $z \neq w$.
Then $\psi(d(z, w)) > 0$
From (2.2.1), now follows that
$\psi(d(Tz, Tw)) \leq \beta(\psi(d(z, w)))\psi(d(z, w)))$
$< \psi(d(z, w))$.
Hence $\psi(d(z, w)) < \psi(d(z, w))$,
a contradiction. Therefore $z = w$.
Therefore T has unique fixed point in X.
Now we have the following Corollary.
Corollary 2.3. Let (X, d) be a complete metric space and T be a selfmap on X. Assume that there exist $\psi \in \Psi$ and $k \in [0, 1)$ such that

$$\psi(d(Tx, Ty)) \leq k\,\psi(d(x, y)) \qquad (2.3.1)$$

Let $x_0 \in X$ and set $x = = Tx_{n-1}$ for $n = 1, 2, 3...$.
Suppose that there exist $\psi \in \Psi$ and $\beta \in S$ such that for all $n, m \in N$.

$$\psi(d(Tx_n, Tx_m)) \leq \beta(\psi(d(x_n, x_m)))\psi(d(x_n, x_m)). \qquad (2.3.2)$$

Then $\{x_n\}$ is Cauchy, $\{x_n\}$ converges to z (say) and z is a unique fixed point of T in X.
Proof. Under condition (2.3.1) we first show that T is continuous. Suppose $\{x_n\}$ converges to x and Tx_n does not converge to Tx. Then there exists an $\varepsilon > 0$ such that $\varepsilon < d(Tx_n, Tx)$ for infinitely many n.
By (2.3.1), $\psi(\varepsilon) \leq \psi(d(Tx_n, Tx)) \leq k\,\psi(d(x_n, x))$ for infinitely many n. Then $\varepsilon < d(x_n, x)$ for infinitely many n. Therefore $\varepsilon = 0$,
a contradiction and hence T is continuous.
As a consequence of Theorem 2.2, we obtain the following Corollary, which is proved in Theorem 1.8 by using the additional hypothesis that ψ is continuous and sub additive.
Corollary 2.4. (Sastry et al. [8]) Let $(X, \preccurlyeq)$ be a partially ordered set and (X, d) be a complete metric space. Let $T: X \to X$ be a continuous non-decreasing mapping such that there exists
$x_0 \in X$ with $x_0 \preccurlyeq T(x_0)$. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$, such that

$$\psi(d(T(x), T(y))) \leq \beta(\psi(d(x, y)))\,\psi(d(x, y)) \qquad (2.4.1)$$

for all $x, y \in X$, whenever x and y are comparable.
Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$. then $\{x_n\}$ is a Cauchy sequence, $\{x_n\}$ converges to z (say) and z is a unique fixed point of T in X.
Proof. Let $x_0 \in X$.
We define $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for $n = 0, 1, 2...$
Since $x_0 \preccurlyeq Tx_0$ and T is a non-decreasing function,
$x_0 \preccurlyeq Tx_0 \preccurlyeq Tx_1 \preccurlyeq Tx_2 \preccurlyeq ...$
i.e., $x_0 \preccurlyeq x_1 \preccurlyeq x_2 ... \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq ...$
so that $x_n \preccurlyeq x_{n+1}$ for each $n \geq 0$.
Thus any two numbers of the sequence $\{x_n\}$ are comparable. Hence by (2.4.1) we have,
for all $n,m \in N$.
$\psi(d(Tx_n, Tx_m)) \leq \beta(\psi(d(x_n, x_m)))\psi(d(x_n, x_m))$
Thus (2.2.1) is satisfied. Hence by Theorem 2.2 $\{x_n\}$ is a Cauchy sequence and hence converges to z (say).
Now, by the continuity of T again by Theorem 2.2 follows that z is a unique fixed point of T.

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